

Part I: Capacities and coefficients

We are all interested in symplectic manifolds. More generally, it is interesting to understand the relations between geometric objects, and a particularly interesting question concerns symplectic embeddings. Some basic ideas go back to non-squeezing [1] and capacities [2], which measure domains by comparing them with standard objects such as balls and cylinders.

We will restrict attention to star-shaped domains in $\mathbb{C}^n$, although the framework is more general. Let X be such a Liouville domain, and let

$$H: S^1 \times \widehat{X} \longrightarrow \mathbb{R}$$

be an admissible Hamiltonian on its completion. We can define the Floer homology of H [3]. Let me briefly remind you that it is the homology of a chain complex whose generators are one-periodic Hamiltonian orbits, or critical points of the action functional. The differential counts zero-dimensional moduli spaces of Floer cylinders from one orbit to another, modulo translation.

Take Hamiltonians that are linear at infinity. There are continuation maps between suitable Hamiltonians with increasing slopes. Taking the direct limit over a monotone cofinal family of Hamiltonians gives symplectic homology [4]. Throughout, we use the grading of Gutt and Hutchings [5]: a constant orbit corresponding to a degree- p generator of the relative Morse complex has Floer degree $p - n$. Thus

$$SH_*(X; R) = \varinjlim_{\text{slope}(H) \rightarrow \infty} HF_*(H; R).$$

A useful way to obtain quantitative invariants is to equip an invariant with a filtration. In this case, each orbit carries an action:

$$\mathcal{A}_H(\gamma) = \int_0^1 \lambda(\dot{\gamma}(t)) dt - \int_0^1 H_t(\gamma(t)) dt.$$

This defines filtered symplectic homology,

$$SH_*^{\leq L}(X; R).$$

Choose $0 < \epsilon < T_{\min}$, where $T_{\min}$ is the least positive action of a closed Reeb orbit on ∂X (necessarily a simple orbit). In the normalized construction, the constant-orbit actions tend to zero; choose the small Hamiltonians so that their constant-orbit sector lies below ϵ . This low-action part, denoted $SH^{\leq \epsilon}$, recovers relative Morse homology with its grading shift:

$$SH_j^{\leq \epsilon}(X; R) \simeq H_{j+n}(X, \partial X; R).$$

For our domains, X is contractible, so the relative homology has one generator, the fundamental class $[X] \in H_{2n}(X, \partial X; R)$, corresponding to a low-action Floer class of degree n .

There is also an S^1 -equivariant version [6], reflecting rotation of loops and using the Borel, or homotopy, quotient. Even for a trivial action, this quotient is a product with BS^1 . The low-action term is therefore

$$SH_j^{S^1, \leq \epsilon}(X; R) \simeq \bigoplus_{a+b=j+n} H_a(X, \partial X; R) \otimes_R H_b(BS^1; R).$$

For a star-shaped domain, this group is concentrated in degrees $n, n+2, n+4, \dots$. Write

$$\alpha_k = [X] \otimes \beta_k \in SH_{n+2k}^{S^1, \leq \epsilon}(X; R), \quad \beta_k \in H_{2k}(BS^1; R), \quad k \geq 0,$$

using the low-action identification above.

There is a map from the low-action part into the L -filtered part:

$$I_L: SH_j^{S^1, \leq \epsilon}(X; R) \longrightarrow SH_j^{S^1, \leq L}(X; R).$$

The capacity is the first level at which the class is killed:

$$c_{k+1}^R(X) = \inf\{L > \epsilon \mid I_L(\alpha_k) = 0\}.$$

For these domains, unfiltered equivariant symplectic homology vanishes, and exactness identifies this definition with detection through the boundary map from positive filtered symplectic homology. The definition is independent of the sufficiently small cutoff ϵ . In particular, the fifteenth capacity uses α_{14} , of Floer degree $n + 28$.

The main geometric property is monotonicity [5]: a generalized Liouville embedding $X \hookrightarrow Y$ implies

$$c_k^R(X) \leq c_k^R(Y) \quad \text{for every } k \geq 1.$$

For star-shaped domains in $\mathbb{C}^n$, every symplectic embedding satisfies this condition, so these capacities obstruct symplectic embeddings.

The original definition of these capacities is due to Gutt and Hutchings [5], who use rational coefficients. Rational coefficients have advantages, but the choice of coefficients matters.

Our first result for today is the following [7]. There are two domains X and Y whose rational capacities and volumes satisfy

$$c_k^{\mathbb{Q}}(X) \leq c_k^{\mathbb{Q}}(Y) \quad \text{for every } k \geq 1, \quad \text{vol}(X) < \text{vol}(Y),$$

but whose fifteenth capacities over $\mathbb{Z}/243$ satisfy the opposite inequality.

The moment regions of these domains are shown in Figure 1. With

$$\mu(z_1, z_2) = (\pi|z_1|^2, \pi|z_2|^2), \quad X_\Omega = \mu^{-1}(\Omega),$$

we take $X = X_{\Omega_X}$ and $Y = X_{\Omega_Y} = E(1, 3)$, where

$$\begin{aligned} \Omega_X &= \{(x, y) \in \mathbb{R}_{\geq 0}^2 \mid x \leq 1, x + y \leq 4/3\}, \\ \Omega_Y &= \{(x, y) \in \mathbb{R}_{\geq 0}^2 \mid 3x + y \leq 3\}. \end{aligned}$$

The capacities of these corner domains are defined by approximation.

In this example,

$$\text{vol}(X) = \frac{5}{6} < \frac{3}{2} = \text{vol}(Y), \quad c_{15}^{\mathbb{Z}/243}(X) = 10 > 9 = c_{15}^{\mathbb{Z}/243}(Y).$$

Consequently, there is no symplectic embedding $X \hookrightarrow Y$, although the rational Gutt–Hutchings capacities [5] and volume give no obstruction.

Part II: Beyond ordinary homology

Consider $S^2 \vee S^4$ and $\mathbb{C}\mathbb{P}^2$. Their homology groups agree with any coefficients, but their stable homotopy groups distinguish them (see [8, Chapter 4]). This comparison concerns homology groups: their cohomology rings already differ.

Here is another example, more directly related to capacities. We can define a toy invariant of filtrations. Take a filtration starting with S^2 , and, for simplicity, assume it has two steps. For a class α in its homology, ask at which filtration level α vanishes. In reduced homology, S^2 has just one generator, which we denote by α . There are only two filtration levels; if the class never vanishes, the invariant is ∞ .

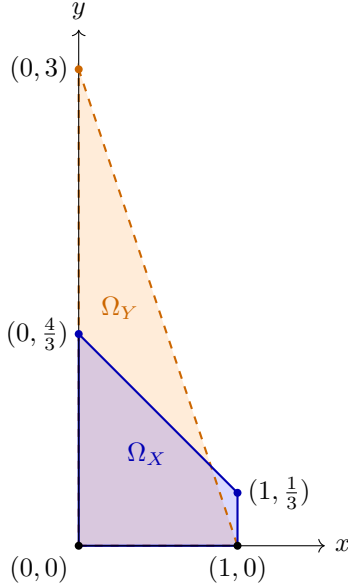


Figure 1: Overlaid moment regions of X (blue, solid boundary) and Y (orange, dashed boundary), with the same scale on both axes.

Consider

$$F_1 = S^2 \hookrightarrow F_2 = \mathbb{CP}^2, \quad G_1 = S^2 \hookrightarrow G_2 = S^2 \vee S^4.$$

The first map is the standard inclusion of $\mathbb{CP}^1 = S^2$, and the second is the inclusion of the S^2 wedge summand. At the level of ordinary homology, the toy capacity is ∞ in both cases: the generator

$$\alpha \in \tilde{H}_2(S^2; \mathbb{Z})$$

never vanishes.

At the level of stable homotopy, however, consider

$$\pi_3^s(S^2) \simeq \mathbb{Z}/2,$$

generated by the stable Hopf class η . In the $\mathbb{CP}^2$ filtration, this class vanishes at level 2. In the wedge filtration, it never vanishes. Thus

$$c_\eta(F) = 2, \quad c_\eta(G) = \infty.$$

This example shows why we might want to retain the filtered homotopy type, before extracting numerical invariants from it.

The idea goes back to Cohen, Jones, and Segal [9]. The Floer differential counts isolated trajectories. A Floer homotopy type also retains information from the higher-dimensional trajectory spaces, their framings, and the way their boundary faces fit together. Its ordinary homology recovers Floer homology. The object is a spectrum: informally, a homotopy type in which suspension is invertible.

In Morse theory, this construction recovers $\Sigma^\infty M_+$. In Floer theory, there need not be an underlying space specified in advance. Generalized homology theories let us examine the resulting object in different ways, including the stable homotopy information illustrated by the Hopf example.

This viewpoint organizes Floer maps and compositions, including suitable derived orbifold moduli spaces [10], and has applications to generalized cuplength bounds for Lagrangian intersections [11] and characteristic- p periodic-orbit bounds via Morava K -theory [12].

The geometric data consist of generators and compactified trajectory spaces between them, with compatible stable framings. Their boundary faces are products of trajectory spaces: they describe a trajectory breaking at an intermediate generator. We call this structure a framed flow category. Figure 2 illustrates its trajectory spaces and broken endpoints for a Morse function on the sphere.

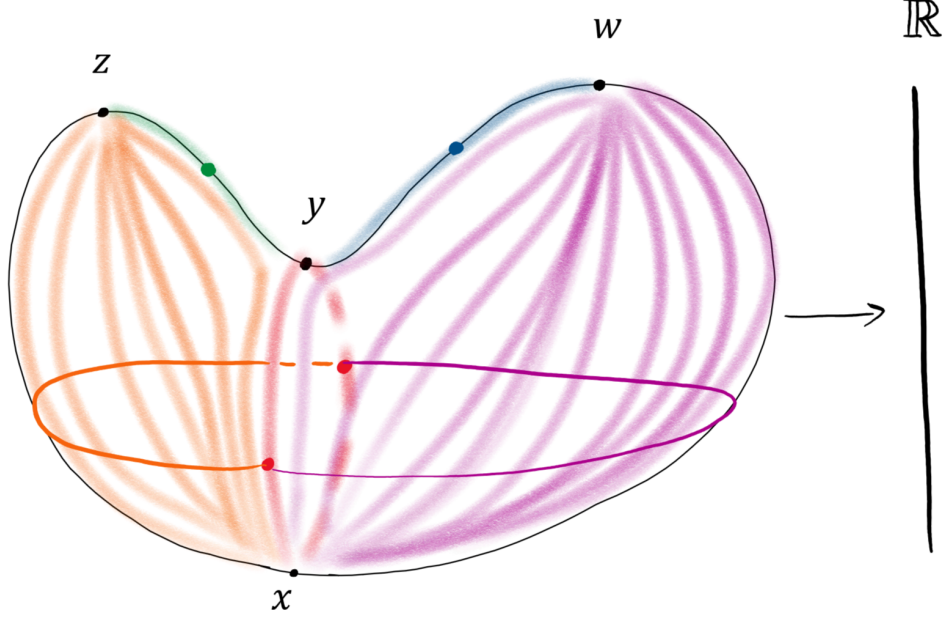


Figure 2: A Morse sphere with $|x| = 0$, $|y| = 1$, and $|z| = |w| = 2$. Solid colored dots and curves represent trajectory spaces and their broken endpoints.

The Pontryagin–Thom construction turns framed trajectory manifolds into attaching maps, and their bordisms and boundary faces into homotopies and higher compatibility data [13, 14].

For symplectic homology, Côté and Kartal [15, Section 5] construct an S^1 -equivariant Floer spectrum under a stable symplectic trivialization hypothesis. The circle action reflects rotation of loops, and Hamiltonian action supplies the filtration.

Our result lifts the capacity construction to this filtered spectrum [16]. Let X be a Liouville domain equipped with the required stable symplectic trivialization, and let E represent a generalized homology theory, such as stable homotopy, bordism, or topological K -theory. For a class

$$\alpha \in E_{j+n}^{S^1}(X, \partial X),$$

we define

$$c_\alpha^E(X)$$

as the first action level at which the corresponding low-action class vanishes in the full filtered theory, just as in Part I; a relative class of degree $j + n$ corresponds to a Floer class of degree j .

We prove monotonicity under generalized Liouville embeddings compatible with the chosen stable symplectic trivializations and corresponding relative classes, so these capacities give embedding obstructions; for star-shaped domains in $\mathbb{C}^n$, we use the standard trivialization. Ordinary rational homology, represented by $E = H\mathbb{Q}$, recovers the usual capacities:

$$c_{[X]_{\mathbb{Q}} \otimes \beta_{k-1}}^{H\mathbb{Q}}(X) = c_k^{\mathbb{Q}}(X), \quad k \geq 1.$$

Here $E_*^{S^1}(X, \partial X)$ is Borel equivariant generalized homology with trivial circle action, and the construction uses a low-action identification and transfer maps compatible with ordinary symplectic homology.

Part III: Computing directly from geometry

We now work in Morse theory. Let M be a closed manifold, equipped with a Morse function f and a Morse–Smale metric. Its framed trajectory spaces recover $\Sigma^\infty M_+$. Our goal is to calculate the stable homotopy groups of this spectrum directly from those spaces, in analogy with the Morse complex for ordinary homology.

To explain our result, we need a computational tool called a spectral sequence. It consists of a bigraded abelian group on each page, together with differentials and a rule for passing to the next page.

The differential d_r sends total degree n and filtration degree p to $(n-1, p-r)$, and the next page is $E_{r+1} = H(E_r, d_r)$.

In our case, the spectral sequence stabilizes after finitely many pages. The entries in the n th column give, up to an extension problem, the group we want to calculate.

First, the construction gives a spectral sequence whose first page is

$$E_1^{n,p} = \bigoplus_{\substack{x \in \text{Crit}(f) \\ \text{ind}_f(x)=p}} \Omega_{n-p}^{\text{fr}} \implies \pi_n(\Sigma^\infty M_+).$$

Here Ω_j^{fr} is the j -dimensional framed bordism group.

Our theorem [17] describes the differentials geometrically, by filling and gluing the framed trajectory manifolds.

Along $n = p$, the first differential gives the ordinary Morse complex; along $n - p = q$, it gives the Morse complex with coefficients in Ω_q^{fr} .

The first insight from this process is that $\mathbb{Z}$ was not just $\mathbb{Z}$: it was the zero-dimensional framed bordism group,

$$\mathbb{Z} = \Omega_0^{\text{fr}}.$$

For those who want a short reminder, a framing on an m -dimensional manifold M , usually called a stable framing—but we will omit the word *stable*—is a choice of bundle isomorphism

$$TM \oplus \mathbb{R}^N \simeq \mathbb{R}^{N+m}.$$

The k -dimensional framed bordism group is the set of closed k -dimensional framed manifolds up to framed bordism: two manifolds are bordant if together, with the appropriate boundary framings, they form the boundary of a compact framed manifold. Disjoint union is the group operation:

$$[M] + [N] = [M \sqcup N].$$

For critical points j and i , write $\mathcal{M}_{j,i}$ for the compactified space of unparametrized gradient trajectories from j to i . These are the framed flow manifolds introduced in Part II.

For illustration, take $\mathbb{RP}^3$ with its standard Morse function, with one critical point in each index from 0 to 3. We now turn the page, which means that we take homology. Let's look at the point of index three. We will describe the second differential on its representative, producing a one-dimensional framed bordism class. Represent the class at index three by $N = \{\text{pt}\}$. Since it survives the first differential, $N \times \mathcal{M}_{3,2}$ is framed null-bordant. This means that it is the boundary of a framed manifold of one dimension higher, which we choose and denote by W :

$$\partial W \simeq N \times \mathcal{M}_{3,2}.$$

Now look at $W \times \mathcal{M}_{2,1}$. Its boundary agrees with that of $N \times \mathcal{M}_{3,1}$, because the latter flow manifold is compactified by adding broken trajectories:

$$\partial \mathcal{M}_{3,1} \simeq \mathcal{M}_{3,2} \times \mathcal{M}_{2,1}.$$

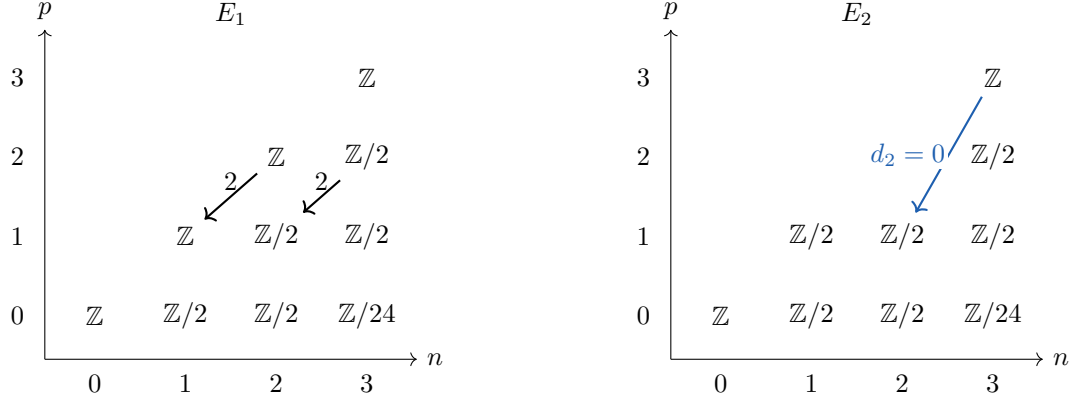


Figure 3: The first two pages for $\mathbb{RP}^3$ with sphere coefficients, in total degree n and filtration degree p . The black arrows are multiplication by 2; the arrow between $\mathbb{Z}/2$ groups is therefore zero. Other first differentials vanish and are omitted. The blue arrow is the second differential described by the gluing construction. Entries with $n < p$ are left blank; only $0 \leq n, p \leq 3$ are shown.

This is also why the square of the Morse differential is zero.

We can therefore glue these two manifolds with boundary along their common boundary. We obtain a closed one-dimensional framed manifold:

$$(N \times \mathcal{M}_{3,1}) \cup_{N \times \mathcal{M}_{3,2} \times \mathcal{M}_{2,1}} (W \times \mathcal{M}_{2,1}).$$

With the prescribed gluing framing, this manifold represents our second differential, interpreted on the second page.

In Figure 4, the upper point is the coefficient source; the arrows record N , the chosen filling W , and the flow manifolds.

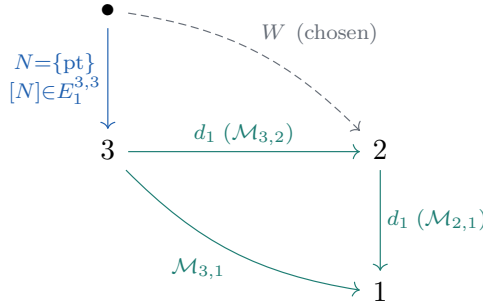


Figure 4: The pieces defining $d_2[N]$.

In this example the glued circle is framed null-bordant, so the second differential vanishes.

In general, changing the representative or filling alters the glued class by an incoming d_1 -boundary, so it defines a class on E_2 independent of these choices.

This illustrates the geometric differential formula: the spectral sequence calculates stable homotopy from critical points and framed flow manifolds, in analogy with Morse homology.

As you can see, the first page of the spectral sequence is built from framed bordism groups. These are ordinary abelian groups, and the positive-degree groups are finite [18]:

$$\Omega_0^{\text{fr}} \simeq \mathbb{Z}, \quad \Omega_j^{\text{fr}} \text{ is finite for every } j > 0.$$

Every finite abelian group is a direct sum of finite cyclic groups. Thus, a natural starting point for calculating these spectral generalizations of symplectic capacities is to study symplectic homology with integral and cyclic coefficients. This brings us back to the beginning of the talk.

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