Morava K-Theory and Equivariant Duality on Global Kuranishi Charts

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12 May 2026

Global Kuranishi charts reading seminar. These notes were prepared with help from GPT.

The starting point is the additive splitting theorem of Abouzaid, McLean, and Smith [1, Theorem 1.1]. We discuss the duality used to define a pushforward in Morava K-theory and how to pass from those coefficients to integral cohomology.

1 The splitting problem

Let (X, ω) be a closed symplectic manifold and let $\phi\colon S^1 \to \mathrm{Ham}(X, \omega)$ be a Hamiltonian loop. Clutching along ϕ gives a bundle $P_\phi \to S^2$ with fiber X. The theorem gives an isomorphism of graded abelian groups

$$H^*(P_\phi; \mathbb{Z}) \cong H^*(S^2; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(X; \mathbb{Z}).$$

It suffices to split the restriction $\rho\colon H^*(P_\phi; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ to a fiber. Indeed, $P_\phi \setminus X$ is a product bundle over an open disk, and restriction to that complement identifies with ρ after identifying its cohomology with that of a nearby fiber. The long exact sequence of the pair then splits into short exact sequences. The relative term $H^*(P_\phi, P_\phi \setminus X; \mathbb{Z}) \cong H^{*-2}(X; \mathbb{Z})$ comes from the Thom isomorphism for the normal bundle of the fiber.

The correspondence. The construction in [1, §§3.2–3.4] uses a degeneration with total space $\widetilde{P}$. The central fiber is $P_\phi \cup_X P_{\phi^{-1}}$, and a fiber over a regular parameter ∞ is $P_\infty \cong S^2 \times X$. Consider genus-zero stable maps in the class of $S^2 \times \{x\}$ in P_∞, with two marked points. One point evaluates to a horizontal copy of $S^2 \times X$ in $\widetilde{P}$, from which we can project to X. Restricting the other evaluation to $P_\bullet$, for $\bullet = \phi, \infty$, gives the correspondence

$$X \xleftarrow{a_\bullet} \mathcal{M}_\bullet \xrightarrow{b_\bullet} P_\bullet.$$

If the moduli space were a closed oriented manifold of the expected dimension, the map from $H^*(X)$ to $H^*(P_\bullet)$ would be

$$\alpha \longmapsto \mathrm{PD}_{P_\bullet}^{-1}((b_\bullet)_*(a_\bullet^*\alpha \frown [\mathcal{M}_\bullet])).$$

At the product fiber the curves are constant in the X-direction. There b_∞ identifies the moduli space with P_∞, and the map is pullback along $P_\infty \to X$. Its restriction to X is the identity. Compatibility of the pushforwards with restriction from the total family transports this identity to P_ϕ. The virtual pushforwards that implement the comparison are constructed in [1, §§5.7–5.9].

2 Duality on the thickening

A smooth global Kuranishi chart presents a moduli space M as Z/G, where $Z = s^{-1}(0)$ for a G-equivariant section s of a vector bundle $E \to U$. The compact Lie group G acts with finite stabilizers on the smooth thickening U. Write $\mathfrak{g}$ for its adjoint representation and set

$$d = \dim U - \dim G - \operatorname{rank} E.$$

Already over $\mathbb{Q}$, the virtual class acts on $\check{H}^d(M;\mathbb{Q})$ by way of the thickening. In [1, §5.1], Alexander duality on the oriented rational homology manifold U/G first gives a homology class supported on M. Passing to equivariant relative homology lets one apply s_* followed by the Thom isomorphism for E. This gives the functional $\check{H}^d(M;\mathbb{Q}) \to \mathbb{Q}$.

Cheng's duality. For the generalized construction, let $\mathbb{K}$ be a complex-oriented ring spectrum that is $K(n)$-local at a fixed prime and height. We use Borel equivariant (co)homology, writing $U_{hG} = EG \times_G U$. For a closed smooth G-manifold U with finite stabilizers, Cheng's duality identifies

$$H_i(U_{hG};\mathbb{K}) \;\cong\; \widetilde{H}^{-i}\big((U_{hG})^{-TU+\mathfrak{g}};\mathbb{K}\big),$$

where the spectrum on the right is the Thom spectrum of the indicated virtual bundle over U_{hG}. Cheng proved the Morava K-theory case [2]. The form above and the version for manifolds with boundary used below are in [1, §§5.6 and 7.2–7.3].

Suppose $-TU+\mathfrak{g}$ is $\mathbb{K}$-oriented. Its virtual rank is $-D$, where $D = \dim U - \dim G$, so the Thom isomorphism turns degree $-i$ on the Thom spectrum into degree $D-i$ on U_{hG}. We can shrink the thickening to a compact invariant neighborhood of Z with smooth boundary, still denoted U, and put $U^c = U/\partial U$. The corresponding duality sends the unit of $H^0_G(U;\mathbb{K})$ to a class

$$[U^c]_{\mathbb{K}} \in \widetilde{H}_D(EG_+ \wedge_G U^c;\mathbb{K}).$$

The orientation of the chart is an orientation of $TU - \mathfrak{g} - E$. After stabilization we can work with compatible orientations of $TU-\mathfrak{g}$ and E, as in [1, Lemma 5.11]. The Thom class of E pulls back along s to a class of degree $\operatorname{rank} E$ in $H^*_G(U, U\setminus Z;\mathbb{K})$. A class on M pulls back to Z and can be represented on a sufficiently small invariant neighborhood, using the continuity property in [1, §5.2]. Multiplying by the pulled-back Thom class puts a class of degree j in degree $j+\operatorname{rank} E$, with support on Z. Since $\partial U \subset U \setminus Z$, the supported class maps to $H^*_G(U,\partial U;\mathbb{K})$. Pairing with $[U^c]_{\mathbb{K}}$ gives a class in $\pi_{d-j}\mathbb{K}$. In particular a degree-d class can be integrated against the virtual class.

Restricting to a smaller neighborhood of Z uses the collapse map on one-point compactifications. Changing the presentation of the orbifold preserves its Borel space and the relevant Thom spectrum. Stabilizing the chart uses compatibility of the duality map with the Thom isomorphism. These are Addenda 5.18–5.19 and Lemma 5.22 of [1]. The pushforward is constructed there in §5.7.

Why locality enters. For a finite group H acting on a $K(n)$-local spectrum A, the norm identifies the localized homotopy orbits $L_{K(n)}(A_{hH})$ with the homotopy fixed points A^{hH}. For the compact Lie group acting on a chart, the norm calculation reduces to finite stabilizers by using cells with orbit type G/H. The Adams isomorphism and equivariant Atiyah duality then identify the two sides with the homology and Thom cohomology in Cheng's theorem. See [1, §§7.2–7.3] for the construction.

3 From Morava K-theory to integral cohomology

For each prime p and $n, k \geq 1$, the theory $K_{p^k}(n)$ is complex-oriented and has coefficients

$$\pi_* K_{p^k}(n) = (\mathbb{Z}/p^k)[v_n^{\pm 1}], \qquad |v_n| = 2(p^n - 1).$$

It is built by successive extensions of $K_p(n)$, so it is $K_p(n)$-local [1, Proposition 5.14 and §7.3]. The push–pull construction thus gives a section of the fiber restriction with all these coefficients. The following argument explains why these sections suffice.

Splitting lemma ([1, Lemma 5.15])**.** *Let $f\colon X \to Y$ be a map of compact manifolds. If*

$$f^*\colon H^*(Y; K_{p^k}(n)) \to H^*(X; K_{p^k}(n))$$

is a split epimorphism for every p, k, n, then $f^*\colon H^*(Y;\mathbb{Z}) \to H^*(X;\mathbb{Z})$ is a split epimorphism of graded abelian groups.

Proof. Fix p and k, and take n so that $N = 2(p^n - 1) > \max(\dim X, \dim Y)$. In the cohomological Atiyah–Hirzebruch spectral sequence, the coefficient degree is a multiple of N. A nonzero differential d_r would require $r \equiv 1 \bmod N$ and $r \geq 2$. There is no room for the corresponding shift by r in the space degree. Moreover, in each total degree between 0 and $\max(\dim X, \dim Y)$, only the coefficient row of degree zero contributes. This gives a natural identification with $H^*(-;\mathbb{Z}/p^k)$ in that range, and hence a section for f^* with $\mathbb{Z}/p^k$-coefficients.

To pass to reductions of integral cohomology, fix a degree i and write $A_V = H^i(V;\mathbb{Z})$, $B_j(V) = H^i(V;\mathbb{Z}/p^j)$, for $V = X, Y$. The coefficient sequence gives

$$0 \longrightarrow A_V/p^j A_V \longrightarrow B_j(V) \longrightarrow H^{i+1}(V;\mathbb{Z})[p^j] \longrightarrow 0,$$

where $[p^j]$ denotes the subgroup killed by p^j. Reduction $B_j(V) \to B_k(V)$ acts on the last term by multiplication by p^{j-k}. For $j - k$ sufficiently large, its image is therefore exactly $A_V/p^k A_V$. Restrict a section at level j to $A_X/p^j A_X$ and then reduce its values to level k. This lands in $A_Y/p^k A_Y$ and factors through $A_X/p^k A_X$, since the target is killed by p^k. Naturality of reduction makes the resulting map a section. By the Chinese remainder theorem there are sections after reduction modulo every positive integer m.

Finally, write $g\colon A_Y \to A_X$ for the integral map. Its finitely generated cokernel has zero reduction modulo every m, so g is surjective. Lift a basis of the free part of A_X. For a torsion cyclic summand generated by an element of order d, choose m divisible by the exponent of the torsion of A_Y and by the square of the exponent of the torsion of A_X. A section modulo m sends the generator to a class killed by d. The free coordinates of this class are multiples of m/d, so their images have zero torsion component in A_X modulo m, using a fixed free–torsion decomposition. The torsion component of the lift is an element of A_Y killed by d and maps to the desired generator. These lifts give an integral section. □

Applying the lemma to the inclusion of a fiber gives the integral section required in Section 1.

References

[1] M. Abouzaid, M. McLean, and I. Smith. *Complex cobordism, Hamiltonian loops and global Kuranishi charts.* arXiv:2110.14320.

[2] M. C. Cheng. *Poincaré duality in Morava K-theory for classifying spaces of orbifolds.* arXiv:1305.2731.